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3 (Sem-1/CBCS) MAT HC 2

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT-HC-1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed :

1×10=10

- (a) Find the polar representation of $z = 2i$.
- (b) If $x = 0$ and $y > 0$, then what is the value of t^* ?
- (c) Write the negation of the statement 'For any integer n , $n^2 > n$ ' in plain English then formulate the negation using set of context and quantifier.

Contd.

- (d) Disapprove the statement using counter example :

"For any $x, y \in \mathbb{R}$, $x^2 = y^2$ implies $x = y$."

- (e) Suppose f is a constant function from X to Y . The inverse image of a subset of Y cannot be

(i) an empty set

(ii) the whole set X

(iii) a non-empty proper subset of X
(Choose the correct option)

- (f) Let $X = Y = \mathbb{R}$. Let $A \subseteq X, B \subseteq Y$. Draw the picture for $A \times B$ where $A = [-1, 1]$ and $B = [2, 3]$.

- (g) Suppose a system of linear equations in echelon form has a 3×5 augmented matrix whose fifth column is a pivot column.

Is the system consistent? Justify.

- (h) If a set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in \mathbb{R}^n contains the $\vec{0}$ vector, is the set linearly independent? Justify.

(i) If $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, compute

$(A\vec{x})^T$.

- (j) What is the determinant of an $n \times n$ elementary matrix E that has been scaled by 7.

2. Answer the following questions : $2 \times 5 = 10$

- (a) If $z = -2\sqrt{3} - 2i$, find the polar radius and polar argument of z .

- (b) Is the function $g: \mathbb{R} \rightarrow \mathbb{R}$ given by

$g(x) = |x - 2|$ one-one and onto?

Explain.

- (c) Let universal set be \mathbb{R} and index set be

\mathbb{N} : For a natural number n , $J_n = \left(0, \frac{1}{n}\right)$.

Identify with justification $\bigcap_{n \in \mathbb{N}} J_n$.

(d) Show that T is a linear transformation by finding a matrix that implements the mapping

$$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$$

(e) A is a 2×4 matrix with two pivot positions. Answer the following with justification :

(i) Does $A\bar{x} = \bar{0}$ have a non-trivial solution?

(ii) Does $A\bar{x} = \bar{b}$ have at least one solution for every \bar{b} ?

Answer **any four** questions from the following : 5×4=20

(a) Find the polar representation of the complex number

$$z = 1 - \cos a + i \sin a \quad a \in [0, 2\pi]. \quad 5$$

(b) Let A and B be subsets of an universal set U . Prove —

$$(i) \quad (A \cap B)^c = A^c \cup B^c$$

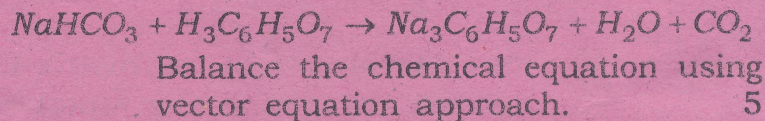
$$(ii) \quad (A \cup B)^c = A^c \cap B^c \quad 5$$

(c) Define bijection.

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be $f(m) = m - 1$, if m is even $f(m) = m + 1$, if m is odd. Show f is a bijection and $f^{-1} = f$. 1+4=5

(d) For vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p \in \mathbb{R}^n$ define $\text{span} \{ \bar{v}_1, \bar{v}_2, \dots, \bar{v}_p \}$ construct a 3×3 matrix A with non-zero elements and a vector \bar{b} on \mathbb{R}^3 such that \bar{b} is not in the set spanned by the columns of A . 2+3=5

(e) Alka-Seltzer contains sodium bicarbonate (NaHCO_3) and citric acid ($\text{H}_3\text{C}_6\text{H}_5\text{O}_7$). When a tablet is dissolved in water the following reaction produces sodium citrate, water and carbon dioxide :



(f) Prove that an $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n , and in this case any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1} . 5

Answer **any four** from the following :
10×4=40

(a) (i) Find the cube roots of the number $z = 1 + i$ and represent them in the complex plane. 5

(ii) Find the number of ordered pairs (a, b) of real numbers such that $(a + ib)^{2002} = a - ib$. 2

(iii) If x, y, z be real numbers such that $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$, prove that $\sin 2x + \sin 2y + \sin 2z = 0$ and $\cos 2x + \cos 2y + \cos 2z = 0$. 3

(b) (i) Solve the equation $z^7 - 2iz^4 - iz^3 - 2 = 0$. 5

(ii) Find the inverse of the matrix if it exists by performing suitable row operations on the augmented matrix $[A : I]$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \quad 5$$

(c) (i) If $f : X \rightarrow Y$ be a map and $B \subseteq Y$, then prove $f^{-1}(B^c) = (f^{-1}(B))^c$. 4

(ii) $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, where $n \in \mathbb{N}$. Find $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$. 2

(iii) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$.

Find $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}([0, 1])$. 4

(d) (i) State the induction principle and use it to show that for any positive integer $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$. 4

(ii) Write as an implication 'square of an even number is divisible by 4'. Then use direct proof to prove it. 3

(iii) Give proof using contrapositive 'For an integer x if $x^2 - 6x + 5$ is even, then x is odd'. 3

(e) (i) Use the invertible matrix theorem to decide if A is invertible

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \\ -5 & -1 & 9 \end{bmatrix} \quad 2$$

(ii) Compute $\det A$ where

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \quad 4$$

(iii) What do you mean by equivalence class for an equivalence relation? For the relation $a \equiv b \pmod{5}$ on \mathbb{Z} , find all the distinct equivalence classes of \mathbb{Z} . 1+3=4

(f) (i) Solve the system of equations

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

(ii) Choose h and k such that the system has

- (a) no solution
 (b) a unique solution
 (c) many solutions

$$\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$$

(iii) Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form. 2

(g) (i) Prove that the indexed set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\vec{v}_1 \neq \vec{0}$, then some \vec{v}_j (with $j > 1$) is a linear combination of the preceding vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{j-1}$. 5



- (ii) Use Cramer's rule to compute the solutions of the system 3

$$-5x_1 + 3x_2 = 9$$

$$3x_1 - x_2 = -5$$

- (iii) Suppose $T: \mathbb{R}^5 \rightarrow \mathbb{R}^2$

and $T(\vec{x}) = A\vec{x}$ for some matrix

A and each \vec{x} in \mathbb{R}^5 .

How many rows and columns does A have? Justify. 2

- (h) (i) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation that rotates each point in \mathbb{R}^2 about the origin through an angle ϕ with the counter-clockwise direction taken as positive. Find the standard matrix for this transformation. 3

- (ii) Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Prove that T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution. 4

- (iii) Find the area of the parallelogram whose vertices are $(0, -2)$, $(6, -1)$, $(-3, 1)$ and $(3, 2)$. 3

